2008-11-04

KOMPUTATIONAL SIP

I LP and SIP

$A \in R^{n \times N}$ $b \in R^N$ $c \in R^n$ $x \in R^N$

P_1: $\min c^T y$ $\quad$ D_1: $\max b^T x$

$A^T y \geq b$ $\quad$ $Ax = c$ $\quad x \geq 0$

P_2 $\max c^T y$ $\quad$ D_2 $\min b^T x$ $\quad x \geq 0$

$A^T y \leq b$ $\quad$ $Ax = c$

Weak duality lemma for P_1, D_1.

Let $A^T y \geq b$ $\quad Ax = c$, then

$$c^T y \geq b^T x$$

Proof $\quad b^T x \leq (A^T y)^T x = y^T A x = y^T c = c^T y$

General duality result for linear systems of equations with invertible matrix A - $\quad A \in R^{n \times n}$, A invertible

Let $Ax = c$ $\quad A^T y = b$

then $b^T x = c^T y$

Proof $\quad b^T x = (A^T y)^T x = y^T A x = y^T c = c^T y$

Optimality condition for the dual pair P_1, D_1

If P_1 and D_1 are both consistent with optimal solutions x and y then we have:

$$Ax = c, \quad x \geq 0$$
$$x^T(A^T y - b) = 0$$
$$A^T y \geq b$$

The simplex algorithm by Dantzig

2008-11-04

<u>Table over states</u>

D_1 ↓ \ P_1 →	BD	UBD	IC
BD	X		
UBD			X
IC		X	X

Under suitable conditions these results may be generalised to SIP.

The pair $\bar{P}_1$ and D_1 may be solved by the simplex algorithm which calls for the solution of a sequence of linear systems of equations. The 3-phase scheme of SIP also leads to a sequence of linear systems of equations.

The main strategy of CSIP is to approximate a given SIP with a task involving a finite number of parameters.

<u>Stieltjes integrals</u> (used to formulate the dual)

Let S be a compact set, C(S) the linear space of continuous functions on S, equipped with the maximum norm. Let L be a continuous linear functional on C(S). Then

$$L(f) = \int_S f(s)\,d\alpha(s), \quad \text{with } \|L\| = \int_S |d\alpha(s)|.$$

Example $$L(f) = \sum_{n=0}^{\infty} \frac{1}{(n+1)^2} f\left(\frac{1}{n+1}\right) = \int_0^1 f(s)\,d\alpha(s)$$

$$\|L\| = \sum_{n=0}^{\infty} \frac{1}{(n+1)^2} = \frac{\pi^2}{6}$$

f may be approximated by a polynomial

$P(a,b,c,S)$: $a(s) \in R^n$, $b(s) \in R$, $c \in R^n$, $y \in R^n$:

$$\inf c^T y$$
$$a(s)^T y \geq b(s), \quad s \in S$$

$D(a,b,c,S)$

$$\max \int_S b(s)\, d\alpha(s)$$
$$\int_S a(s)\, d\alpha(s) = c, \quad d\alpha(s) \geq 0, \; s \in S$$

__Weak__ duality lemma

Let $d\alpha(s) \geq 0$ satisfy

$$\int_S a(s)\, d\alpha(s) = c$$

and $a(s)^T y \geq b(s)$, $s \in S$

then $\int_S b(s)\, d\alpha(s) \leq c^T y$

__Proof__ $\int_S b(s)\, d\alpha(s) \leq \int_S a(s)^T y \cdot d\alpha(s) = \int_S a(s)^T d\alpha(s) \cdot y = c^T y$.

__Optimality__ conditions

Under general assumptions there is an optimal solution $d\alpha$ represented by a finite sum,

$$\int_S a(s)\, d\alpha(s) = \sum_{j=1}^{q} x_j\, a(s_j)$$

such that the vectors $a(s_1), \ldots a(s_q)$ are linearly independent and $x_j > 0$. Then we have the optimality conditions

$$\sum_{j=1}^{q} x_j\, a(s_j) = c \qquad x_j > 0$$
$$x_j\,(y^T a(s_j) - b(s_j)) = 0 \qquad j = 1, \ldots, q$$
$$y^T a(s) \geq b(s) \geq 0 \qquad s \in S$$

$\Rightarrow$ $y^T a(s) - b(s)$ has local minima at s_j, $j = 1, \ldots, q$

Special case:

$$S = [-1, 1]$$

$$\sum_{i=1}^{q} x_i\, a(s_i) = c \qquad x_i > 0$$

$$x_i^T \left(y^T a(s_i) - b(s_i)\right) = 0\,, \quad i = 1, \dots, q.$$

$$(1 - s_i^2)\left(y^T a'(s_i) - b'(s_i)\right) = 0 \quad i = 1, \dots, q$$

<u>Application</u> $\underline{Ex}$ $a : S \to R$ $\quad a \in C(S)$, $b : S \to R$ $\quad b \in C(S)$ $\quad a(s) > 0$

$$\inf\ c\,y \qquad\qquad \max \int_S b(s)\,d\alpha(s) \qquad d\alpha(s) \geq 0$$

$$y\,a(s) \geq b(s) \qquad\qquad \int_S a(s)\,d\alpha(s) = c$$

solution $\quad y = \sup \dfrac{b(s)}{a(s)}$

$$\min \int b(s)\,d\alpha(s) \qquad d\alpha(s) \geq 0$$

$$\sup\ c\,y \qquad\qquad \int a(s)\,d\alpha(s) = c$$

$$y\,a(s) \leq b(s)$$

solution $\quad y = \inf \dfrac{b(s)}{a(s)}$

Optimality condition for $S = [-1, 1]$

$$(1 - s_i^2)\,\frac{d}{ds}\left(\frac{b(s_i)}{a(s_i)}\right) = 0\,, \quad i = 1, \dots, q$$

Two-sided approximation in the maximum norm

$$\min\ y_0$$

$$|a(s)^T y - b(s)| \leq y_0$$

$$\min\ y_0 \qquad\qquad \max \int_S b(s)\,d\alpha(s)$$

$$a(s)^T y + y_0 \geq b(s),\ s \in S \qquad\qquad \int_S a(s)\,d\alpha(s) = 0$$

$$-a(s)^T y + y_0 \geq -b(s),\ s \in S \qquad\qquad \int_S |d\alpha(s)| \leq 1$$

<u>Special case</u> $\quad S = [-1, 1]$

$$a_r(s) = T_{r-1}(s)\,, \quad r = 1, 2, \dots, n.$$

T_r is Čebyšev-polynomial.

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<u>Moment cone</u>

$$M_n = \left\{ \gamma \;\middle|\; \gamma_r = \int_S a_r(s)\, d\alpha(s),\ d\alpha(s) \geq 0,\ r = 1, 2, \dots, n \right\}$$

$$M_{n+1} = \left\{ \gamma \;\middle|\; \gamma_r = \int_S a_r(s)\, d\alpha(s),\ d\alpha(s) \geq 0,\ r = 1, 2, \dots, n+1 \right\}$$

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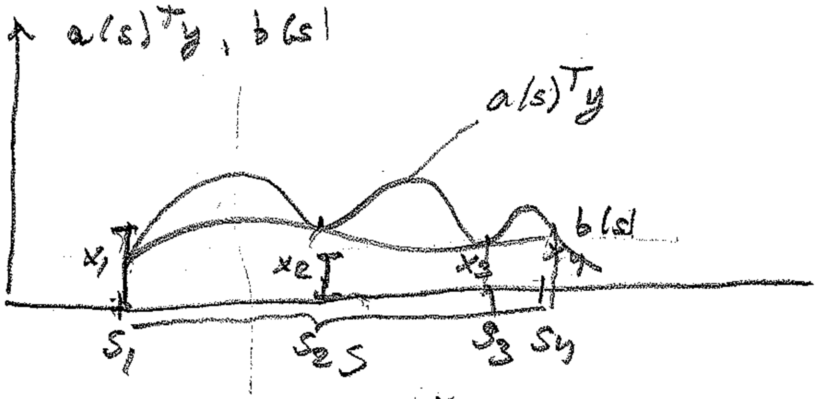
$P(a, b, c, S)$ as one-sided approximation problem



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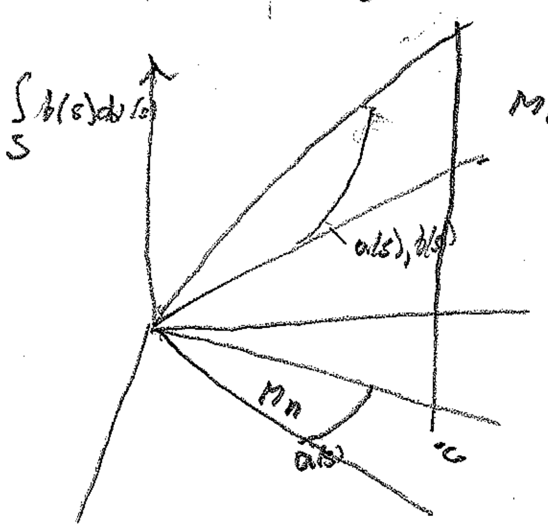
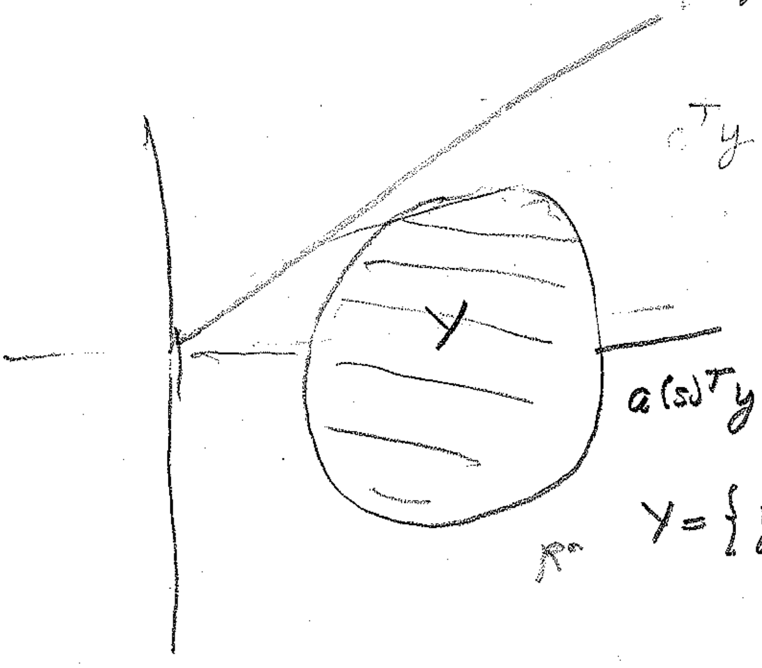




$a(s)^T y \geq b(s),\ s \in S$

$Y = \{ y \in R^n \mid a(s)^T y \geq b(s),\ s \in S \}$

<u>Def</u> $a_1, \dots, a_n$ satisfy Krein's condition if there is $z \in R^n$, such that $z^T a(s) > 0$, $s \in S$

<u>Def</u> $P(a,b,c,S)$ satisfy Slater's condition if there is $z \in R^n$ such that

$$z^T a(s) - b(s) > 0, \quad s \in S$$

Theorem: The dual pair $P(a,b,c,S)$ and $D(a,b,c,S)$ has the same duality states as two linear programs if

1) $c \in M_n^0$, the interior of the moment cone
2) Slater's condition is met.

<u>Ex</u> $\inf cy$ $\qquad$ $\max \int_S b(s)\,d\alpha(s)$

$y\,a(s) \geq b(s)$, $s \in S$ $\qquad$ $\int_S a(s)\,d\alpha(s) = c$

$a, b \in C(S)$, S compact $\qquad$ $d\alpha(s) \geq 0$, $s \in S$

Krein's condition $\Rightarrow a(s) > 0$, $s \in S$

Example $a(s) = 1$

Assume $a(s_0) = 0$. Then neither Krein's nor Slater's condition may hold.

Slater's condition $\Rightarrow a(s) \neq 0$ everywhere.

Computer representation of a SIP

We consider a given computer and operating system. Let C_R be the set of real numbers having an exact computer representation. Then C_R is a finite and hence bounded set. Let C_{R+} be the set of nonnegative computer numbers. Then C_{R+} may be written as an increasing sequence

$$0 = x_1 < x_2 < \dots < x_{NR}$$

If $x \in R$ is larger than x_{NR} we may put $x_{NR} = \infty$. Almost all computers are equipped with the IEEE standard.

Let f be a given function defined on R. Then we approximate $f(x)$ by

$$\text{float}(f(\bar{x})), \quad \bar{x} = \text{float}(x)$$

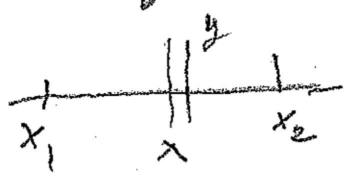
where float(x) is the closest approximation to x in C_R. Thus $P(a, b, c, S)$ is represented by the LP

$$\min \ c^T y$$
$$\text{float}(a(s)^T y) \geq \text{float}(b(s)), \ s \in C_R.$$

Representation of equality. (Definition)

The reals x and y are said to be computationally equal if float(x) and float(y) are the same or adjacent elements in C_R.

Example Let $x_1 < x_2$ be two adjacent members in C_R. Put $x = \frac{x_1 + x_2}{2}$ and set float$(x) = x_1$, breaking the tie in this way. Put $y = x + \varepsilon$, $\varepsilon > 0$. Then float$(y) > x_1$ for all $\varepsilon > 0$. Thus we define computational equality as above.



Verifying feasibility; error propagation

Let $y \in R^n$ be given. We want to verify that y is feasible, i.e.

$$a(s)^T y \geq b(s), \; s \in S.$$

We note that we represent our SIP using a finite set of numbers given in finite precision. For IEEE standard, single precision the relative error is $\leq 0.6 \cdot 10^{-7}$. We need to evaluate for a fixed $s \in C_S$ the expression

$$d(s) = \sum_{r=1}^{n} a_r(s)\, y_r - b(s)$$

We obtain the calculated result:

$$d(s) + \delta d(s) = \sum_{r=1}^{n} (a_r(s) + \delta a_r(s))\,(y_r + \delta y_r) - (b(s) + \delta b(s))$$

subtracting we find

$$\delta d(s) = \sum_{r=1}^{n} (a_r(s)\,\delta y_r + \delta a_r(s) \cdot y_r + \delta a_r(s) \cdot \delta y_r) - \delta b(s)$$

giving the bound

$$|\delta d(s)| \leq \sum_{r=1}^{n} (|a_r(s)\,\delta y_r| + |\delta a_r(s)\, y_r| + |\delta a_r(s)\,\delta y_r|) + |\delta b(s)|$$

Let now

$$|\delta y_r| \leq \varepsilon |y_r| \quad |\delta a_r(s)| \leq \varepsilon |a_r(s)| \quad |\delta b(s)| \leq \varepsilon |b(s)|$$

and neglect the products $\delta a_r(s) \cdot \delta y_r$

Then we obtain

$$|\delta d(s)| \lesssim \varepsilon \left(2\sum_{r=1}^{n} |a_r(s)\, y_r| + |b(s)| \right)$$

Note that the magnitude of the uncertainty $\delta d(s)$ is governed by the magnitude of the coefficients y_r.

Thus if $a_1, a_2, \ldots a_n$ form an orthonormal system then the coefficients y_r are of modest size, since we have

$$\sum_{r=1}^{n} y_r^2 = \|\sum_{r=1}^{n} a_r y_r\|_2^2 = \int_S \left(\sum_{r=1}^{n} y_r a_r(s)\right)^2 \omega(s)\, ds$$

Generalised discretisation

We want to approximate our SIP with a task that can be solved by means of a finite number of arithmetic operations, including verification of optimality of the calculated solution.

Def Positive interpolating operator with nodes $T = \{t_1, \dots, t_N\}$

$$Lf = \sum_{j=1}^{N} \omega_j(s) f(t_j) \quad \text{where}$$

$$\omega_j(s) \geq 0 \quad s \in S$$

$$\omega_j(s_i) = \delta_{ij}$$

Theorem the following two sets of inequalities have identical solution sets y

I $\quad y^T a(s) \geq b(s), \quad s \in T \quad$, T is finite

II $\quad L\, y^T a(s) \geq Lb(s) \quad s \in S$

Discretisation error as perturbation of given problem is represented as approximating a, b with La, Lb.

Example Linear interpolation in one or several dimension.

Numerical example

$S = [-1, 1] \quad \inf\; y_1 + \frac{2}{3} y_3$

$$y_1 + y_2 s + y_3 s^2 + y_4 s^3 \geq e^s.$$

Let T be an equidistant grid with step-size h. Interpolation error R_T for linear interpolation

$$R_T \leq \frac{h^2}{8} \|f''\|$$

The 5 functions $1, s, s^2, s^3, e^s$ have second derivatives bounded by : $0, 0, 2, 3, e$

and hence $R_T \leq \frac{3}{8} h^2 \leq 0.375 \cdot 10^{-6}$ if $h \leq 10^{-3}$ (2000 points)

Piecewise cubic interpolation.

Cubic Hermite interpolation

$$R_f \leq \|f^{(4)}\|_\infty \cdot \frac{h^4}{24\cdot 16} = \|f^{(4)}\|_\infty \frac{h^4}{384} \approx \|f^{(4)}\|_\infty h^4 \cdot 3\cdot 10^{-3}$$

If we take $h = 0.1$ in the preceding example we get
$R_f \leq 7\cdot 10^{-3}$
Feasibility of vector y is verified with a finite number of operations.

Interpolation at the Čebyšev points.

If a_r, $r=1,\dots,n$ and b have an analytic continuation to an ellipse with foci at $+1, -1$, then the interpolating polynomial of degree N at the Čebyšev points approximates a_r with an error decreasing exponentially in N.

Example Global maximisation

$$\min y$$
$$y \geq b(s),\ s\in S,\ b\in C(S).$$

We have for each $s_0 \in S$ $y \geq b(s_0)$. This illustrates the duality lemma since we may take $d\alpha(s_0) = 1$.

For $S\in[-1,1]$ we may approximate b by simpler functions whose maximal value we may calculate by means of a finite number of operations.

If S has many dimensions we may contemplate stochastic approach. This is possible also for general SIP's.

Generalised moment problem.

Let S be compact

$b \in C(S)$ $u_r \in C(S)$, $r = 1, 2, \ldots, n$

Evaluate $\int_S b(s)\,d\alpha(s)$

$$\int_S u_r(s)\,d\alpha(s) = c_r, \quad r = 1, 2, \ldots,$$

when $\int_S |d\alpha(s)| \leq k$

Truncated moment problem:

Evaluate bounds for $\int_S b(s)\,d\alpha(s)$

$$\int_S u_r(s)\,d\alpha(s) = c_r, \quad r = 1, \ldots, n$$

$$\int_S |d\alpha(s)| \leq k$$

We assume that c_r are represented by the computer numbers $\bar{c}_r$ where $|\bar{c}_r - c_r| \leq \varepsilon \|c\|_\infty$ ε = machine-ε

Consider now the problems: Find

$$c_n^M = \max_{d\alpha \geq 0} \int_S u_n(s)\,d\alpha(s)$$

$$\int_S u_r(s)\,d\alpha(s) = c_r, \quad r = 1, 2, \ldots, n-1$$

$$\int_S |d\alpha(s)| \leq k$$

$$c_n^m = \min_{d\alpha \geq 0} \int_S u_n(s)\,d\alpha(s)$$

$$\int_S u_r(s)\,d\alpha(s) = c_r \quad r = 1, 2, \ldots, n-1$$

$$\int_S |d\alpha(s)| \leq k$$

If now $|c_n^M - c_n^m| \leq \varepsilon \cdot \|c_n\|$ then the relation

$\int_S u_n(s)\,d\alpha(s) = c_n$ is redundant and may be removed.

Let $Q_1(t) = y_1^T u(t) \geq u_n(t)$, $t \in S$ $y_1 \in R^{n-1}$

$Q_2(t) = y_2^T u(t) \leq u_n(t)$ $t \in S$ $y_2 \in R^{n-1}$

$\Rightarrow$ $y_1^T c \geq c_n^M \geq c_n^m \geq y_2^T c$ where c_n^M, c_n^m are max and min when $d\alpha$ varies over positive measures

Example 1 $S = [0,1]$ $u_r(t) = t^{r-1}$, $r = 1, 2, \ldots, n, n+1$

Q interpolates at the shifted Cebyser points

$$|u_{n+1} - Q(t)| \leq \frac{2 \cdot \|u_n^{(n)}\|}{4^n \cdot n!} = 2^{2n-1} = 10^{-0.3(2n-1)}$$

Example 2 $S = [0,1]$ u_r as in example 2

$$|u_{n+1} - Q(t)| \leq \frac{2 \cdot 2^{-n}}{n!} \|u_n^{(n)}\|_\infty = 2^{n-1} = 10^{-0.3(n-1)}$$

when Q interpolates at the Chebyshev points

Example 3 $S = [-1, 1]$ $u_r(t) = T_r(t)$ where as before T_r are the Chebyshev polynomials,

$$T_0(t) = 1 \quad T_1(t) = t \quad T_r(t) = 2t \cdot T_{r-1}(t) - T_{r-2}(t)$$
$$r = 2, 3, \ldots$$

Then $\|u_{n+1} - Q\|_\infty = 1$

when Q interpolates at the Chebyshev points.

The relation condition

$$\int_S u_n(t)\, d\alpha(t) = c_n$$

becomes redundant and should be removed if u_n is a linear combination of $u_1, u_2, \cdots u_{n-1}$, and c_n is the same linear of $c_1, c_2, \cdots c_{n-1}$, i.e.

$$u_n = \sum_{r=1}^{n-1} y_r u_r \quad , \quad c_n = \sum_{r=1}^{n-1} y_r c_r$$

Lobatto and Gauss rules

$$c_r = \int_{-1}^{1} t^{r-1}\,dt \qquad \text{Lobatto}$$

$c_1 = 2 \quad c_2 = 0 \quad c_3 = 2/3 \quad c_4 = 0 \quad c_5 = 2/5 \quad c_6 = 0$

<u>Lobatto</u> 4-masspoints $s_1 = -1 \quad s_4 = 1$

$$
\begin{aligned}
x_1 + x_2 + x_3 + x_4 &= 2 \\
-x_1 + s_2 x_2 + s_3 x_3 + x_4 &= 0 \\
x_1 + s_2^2 x_2^2 + s_3^2 x_3 + x_4 &= 2/3 \\
-x_1 + s_2^3 x_2 + s_3^3 x_3 + x_4 &= 0 \\
x_1 + s_2^4 x_2 + s_3^4 x_3 + x_4 &= 2/5 \\
-x_1 + s_2^5 x_2 + s_3^5 x_3 + x_4 &= 0
\end{aligned}
$$

add consecutive equations

$$
\begin{aligned}
\Rightarrow x_1 + x_2 + x_3 + x_4 &= 2 \\
x_2(1+s_2) + x_3(1+s_3) + 2x_4 &= 2 \\
x_2 s_2(1+s_2) + x_3 s_3(1+s_3) + 2x_4 &= 2/3 \\
x_2 s_2^2(1+s_2) + x_3 s_3^2(1+s_3) + 2x_4 &= 2/3 \\
x_2 s_2^3(1+s_2) + x_3 s_3^3(1+s_3) + 2x_4 &= 2/5 \\
x_2 s_2^4(1+s_2) + x_3 s_3^4(1+s_3) + 2x_4 &= 2/5
\end{aligned}
$$

subtract consecutive equations

$$
\begin{aligned}
x_1 + x_2 + x_3 + x_4 &= 2 \\
x_2(1+s_2) + x_3(1+s_3) + 2x_4 &= 2 \\
x_2(1-s_2^2) + x_3(1-s_3^2) &= 4/3 \\
x_2(1-s_2^2)s_2 + x_3(1-s_3^2)s_3 &= 0 \\
x_2(1-s_2^2)s_2^2 + x_3(1-s_3^2)s_3^2 &= 4/15 \\
x_2(1-s_2^2)s_2^3 + x_3(1-s_3^2)s_3^3 &= 0
\end{aligned}
$$

Put $x_2(1-s_2^2) = \xi_2$

$x_3(1-s_3^2) = \xi_3$

$$\xi_2 + \xi_3 = 4/3$$
$$\xi_2 s_2 + \xi_3 s_3 = 0$$
$$\xi_2 s_2^2 + \xi_2 s_3^2 = 4/15$$
$$\xi_2 s_2^3 + \xi_3 s_3^3 = 0$$

Let $w_1 + w_2 s + s^2$ be a polynomial with roots in s_2 and s_3. We determine w_1 and w_2

$\frac{4}{3} w_1 + 0 \cdot w_2 + \frac{4}{15} = 0 \qquad \frac{4}{3} w_1 + \frac{4}{15} = 0$

$0 \cdot w_1 + \frac{4}{15} w_2 + 0 = 0 \qquad \frac{4}{15} w_2 = 0$

$\therefore \; w_1 = -\frac{1}{5} \qquad w_2 = 0$

$-\frac{1}{5} + s^2 = 0 \qquad s_2 = -\frac{1}{\sqrt{5}} \quad s_2 = \frac{1}{\sqrt{5}}$

$$\left.\begin{aligned} \xi_2 + \xi_3 &= 4/3 \\ \xi_2 - \xi_3 &= 0 \end{aligned}\right\} \Rightarrow \xi_2 = 2/3 \quad \xi_3 = 2/3$$

$x_2 = \frac{2}{3} \frac{1}{1-\frac{1}{5}} = \frac{2 \cdot 5}{3 \cdot 4} = \frac{5}{6}$

$x_3 = \frac{5}{6}$

$\therefore \quad x_1 = \frac{1}{6} \qquad s_1 = -1$

$x_2 = 5/6 \qquad s_2 = -\frac{1}{\sqrt{5}}$

$x_3 = 5/6 \qquad s_3 = \frac{1}{\sqrt{5}}$

$x_4 = \frac{1}{6} \qquad s_4 = 1$

<u>3 Gauss</u>

$$x_1 + x_2 + x_3 = 2$$
$$x_1 s_1 + x_2 s_2 + x_3 s_3 = 0$$
$$x_1 s_1^2 + x_2 s_2^2 + x_3 s_3^2 = 2/3$$
$$x_1 s_1^3 + x_2 s_2^3 + x_3 s_3^3 = 0$$
$$x_1 s_1^4 + x_2 s_2^4 + x_3 s_3^4 = 2/5$$
$$x_1 s_1^5 + x_2 s_2^5 + x_3 s_3^5 = 0$$

Let $Q(s) = w_1 + w_2 s + w_3 s^2 + s^3$ be a polynomial with roots in s_1, s_2 and s_3

$$2w_1 + 0\cdot w_2 + \tfrac{2}{3} w_3 + 0\cdot 1 = 0$$
$$0\cdot w_1 + \tfrac{2}{3} w_2 + 0\cdot w_3 + \tfrac{2}{5} = 0$$
$$\tfrac{2}{3} w_1 + 0\cdot w_2 + \tfrac{2}{5} w_3 + 0 = 0$$

$$2w_1 \quad + \tfrac{2}{3} w_3 = 0$$
$$\tfrac{2}{3} w_2 \quad + \tfrac{2}{5} = 0$$
$$\tfrac{2}{3} w_1 \quad + \tfrac{2}{5} w_3 = 0$$

$\therefore$ $w_1 = 0$ $\quad w_2 = -\frac{3}{5}$ $\quad w_3 = 0$

$Q(s) = -\frac{3}{5} s + s^3$ $\quad Q(s) = 0$ $\therefore$ $s_1 = -\sqrt{3/5}$ $\quad s_2 = 0$

$s_1 = -\sqrt{3/5}$ $\quad s_2 = 0$ & $s_3 = \sqrt{3/5}$ $\quad x_1 = \frac{5}{9}$ $\quad x_2 = \frac{8}{9}$ $\quad x_3 = \frac{5}{9}$

If $f^{(6)}(s) > 0$ $\quad s \in [-1,1]$ Lobatto and Gauss rules give upper and lower bounds for $\int_{-1}^{1} f(s)\,ds$

We have the following results

If $f''(t) > 0, \quad -1 \le t \le 1$

$$f(-1) + f(1) \ge \int_{-1}^{1} f(t)\,dt \ge 2f(0)$$

TRAPEZOIDAL RULE

MIDPOINT RULE

If $f^{(4)}(t) > 0, \quad -1 \le t \le 1$

$$\frac{1}{3}f(-1) + \frac{4}{3}f(0) + \frac{1}{3}f(1) \ge \int_{-1}^{1} f(t)\,dt \ge f(-s_1) + f(s_1)$$

SIMPSON'S RULE

TWO-POINT GAUSS

WE DETERMIN s_1. THE FOLLOWING RELATIONS HOLD

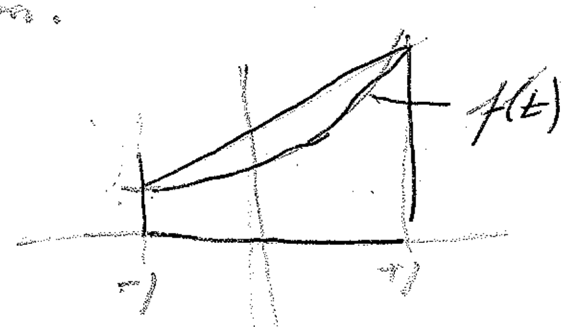
$$2s_1^2 = \frac{2}{3} \qquad s_1 = \sqrt{1/3}$$

Best approximation in the maximum norm.

Let $f \in C^2[-1,1]$ with $f''(s) > 0$, $s \in [-1,1]$

Example $f(s) = e^s$

We seek the straight line $y_1 + y_2 s$ approximates f in the maximum norm.



$$\left.\begin{aligned} y_1 - y_2 + e &= f(-1) \\ y_1 + y_2 + e &= f(1) \end{aligned}\right\} \Rightarrow y_2 = \frac{f(1) - f(-1)}{2}$$

$$y_1 + y_2 s - e = f(s) \qquad \text{Determine } s \text{ from}$$

$$y_2 = f'(s) \qquad y_2 = f'(s)$$

Approximation of f on $[-1,1]$ in the maximum norm with a polynomial of degree $< n$.

$$Q(t) = \sum_{r=1}^{n} y_r T_{r-1}(t)$$

Optimality conditions: There are $n+1$ points t_i $-1 \le t_1 < t_2 < \cdots < t_{n+1} \le 1$ such that

$$|f(t_i) - Q(t_i)| = y_0 \quad i = 1, 2, \ldots, n+1$$

$$(1 - t_i^2)\, Q'(t_i) = f'(t_i)\ , \quad i = 1, 2, \ldots, n+1$$

Evaluation of power series.

Let $c_0, c_1 \ldots$ be such that

$$c_r = \int_0^1 t^r d\alpha(t) \qquad d\alpha(t) \geq 0$$

We want to evaluate the function defined by the power series

$$F(z) = \sum_{r=0}^{\infty} z^r c_r \qquad z = x + iy$$

The analytic continuation is ~~given~~ defined by

$$F(z) = \int_0^1 \frac{1}{1-tz} d\alpha(t)$$

when $\quad \int_0^1 t^r d\alpha(t) = c_r \quad , r = 0, 1, \cdots,$

We note that

$$\frac{1}{1 - tx - ity} = \frac{1 - tx + ity}{(1-tx)^2 + t^2y^2}$$

and hence

$$F(z) = \int_0^1 \frac{1-tx}{(1-tx)^2 + t^2y^2} d\alpha(t) + i \int_0^1 \frac{ty}{(1-tx)^2 + t^2y^2}$$

$$\int_0^1 t^r d\alpha(t) = c_r \quad , r = 0, 1, \cdots, n-1$$

Using SIP we may, for each n, calculate upper and lower bounds the real and imaginary parts of $F(z)$.