Control theory on time scales

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Outline of the talk

1. Motivation.
2. Calculus on time scales
3. Differential equations on time scales.
4. Control systems on time scales - selected examples.
6. Concluding remarks.

Motivation

- Theories of continuous-time control systems and discrete-time control systems consist of similar results.
  For example, controllability of

$$\dot{x} = Ax + Bu \quad \text{and} \quad x^+ = Ax + Bu$$

  where $x^+(t) = x(t+1)$, is given by the same condition:

$$\text{rank}(B, AB, \ldots, A^{n-1}B) = n.$$

- There is need for a common language that would allow for unification of both theories.

- Calculus on time scales unifies theory of differential equations and difference equations (Stefan Hilger, 1988). It can serve as a unifying language for control theory.

- Time scales allow to model systems for which time is partly continuous and partly discrete.

Calculus on time scales

A *time scale* $\mathbb{T}$ is an arbitrary nonempty closed subset of the set $\mathbb{R}$ of real numbers. The standard cases comprise $\mathbb{T} = \mathbb{R}$, $\mathbb{T} = \mathbb{Z}$, $\mathbb{T} = h\mathbb{Z}$ for $h > 0$.

Define
the *forward jump operator* $\sigma : \mathbb{T} \to \mathbb{T}$ by $\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}$;
the *backward jump operator* $\rho : \mathbb{T} \to \mathbb{T}$ by $\rho(t) := \sup\{s \in \mathbb{T} : s < t\}$;
the *graininess function* $\mu : \mathbb{T} \to [0, \infty)$ by $\mu(t) := \sigma(t) - t$.

If $\sigma(t) > t$, we say that t is *right-scattered*, while if $\rho(t) < t$ we say that t is *left-scattered*. If $t < \sup \mathbb{T}$ and $\sigma(t) = t$, then t is called *right-dense*; if $t > \inf \mathbb{T}$ and $\rho(t) = t$ then t is called *left-dense*.

Finally we define the set

$$\mathbb{T}^k := \begin{cases} \mathbb{T} \setminus \{\sup \mathbb{T}\} & \text{if } \rho(\sup \mathbb{T}) < \sup \mathbb{T} < \infty \\ \mathbb{T} & \text{otherwise} \end{cases}$$

Example 1. If $\mathbb{T} = \mathbb{R}$, then for any $t \in \mathbb{R}$, $\sigma(t) = t = \rho(t)$; the graininess function $\mu(t) \equiv 0$.
If $\mathbb{T} = \mathbb{Z}$ then for every $t \in \mathbb{Z}$, $\sigma(t) = t + 1$, $\rho(t) = t - 1$; the graininess function $\mu(t) \equiv 1$.

Definition 2. Let $f : \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}^k$. *Delta derivative* of f at t, denoted by $f^{\triangle}(t)$, is the real number (provided it exists) with the property that given any ε there is a neighborhood $U = (t - \delta, t + \delta) \cap \mathbb{T}$ (for some $\delta > 0$) such that

$$|(f(\sigma(t)) - f(s)) - f^{\triangle}(t)(\sigma(t) - s)| \leq \varepsilon |\sigma(t) - s|$$

for all $s \in U$. Moreover, we say that f is *delta differentiable* on $\mathbb{T}^k$ provided $f^{\triangle}(t)$ exists for all $t \in \mathbb{T}^k$.

Remark 3. If $\mathbb{T} = \mathbb{R}$, then $f : \mathbb{R} \rightarrow \mathbb{R}$ is delta differentiable at $t \in \mathbb{R}$ iff

$$f^{\triangle}(t) = \lim_{s \longrightarrow t} \frac{f(t) - f(s)}{t - s} = f'(t).$$

i.e. iff f is differentiable in the ordinary sense at t.
If $\mathbb{T} = \mathbb{Z}$, then $f : \mathbb{Z} \rightarrow \mathbb{R}$ is always delta differentiable at every $t \in \mathbb{Z}$ with

$$f^{\triangle}(t) = \frac{f(\sigma(t)) - f(t)}{\mu(t)} = f(t+1) - f(t)$$

Example 4. The delta derivative of t^2 is $t + \sigma(t)$.
The delta-derivative of $\frac{1}{t}$ is $\frac{-1}{t\sigma(t)}$.

Proposition 5. *Assume that* $f, g : \mathbb{T} \to \mathbb{R}$ *are delta-differentiable at* $t \in \mathbb{T}^k$ *and* $a \in \mathbb{R}$. *Then:*

$$\begin{aligned}
(f+g)^{\triangle}(t) &= f^{\triangle}(t) + g^{\triangle}(t);\\
(af)^{\triangle}(t) &= af^{\triangle}(t);\\
(fg)^{\triangle}(t) &= f^{\triangle}(t)g(t) + f(\sigma(t))g^{\triangle}(t) =\\
&= f(t)g^{\triangle}(t) + f^{\triangle}(t)g(\sigma(t));\\
\left(\frac{f}{g}\right)^{\triangle}(t) &= \frac{f^{\triangle}(t)g(t) - f(t)g^{\triangle}(t)}{g(t)g(\sigma(t))}.
\end{aligned}$$

Proposition 6. *Let* $g : \mathbb{R} \to \mathbb{R}$ *be differentiable and* $f : \mathbb{T} \to \mathbb{R}$ *be delta differentiable. Then* $g \circ f$ *is delta differentiable and for* $t \in \mathbb{T}$

$$(g \circ f)^{\triangle}(t) = \int_0^1 g'(f(t) + h\mu(t)f^{\triangle}(t))dh \cdot f^{\triangle}(t).$$

A function $f : \mathbb{T} \to \mathbb{R}$ is called *rd-continuous* provided it is continuous at right-dense points in $\mathbb{T}$ and its left-sided limits exist (finite) at left-dense points in $\mathbb{T}$.

It can be shown that
f is continuous $\Rightarrow$ f is rd-continuous.

A function $F : \mathbb{T} \to \mathbb{R}$ is called an *antiderivative* of $f : \mathbb{T} \to \mathbb{R}$ provided $F^{\triangle}(t) = f(t)$ holds for all $t \in \mathbb{T}^k$.
Cauchy integral is defined by

$$\int_r^s f(t) \triangle t = F(s) - F(r) \quad \text{for all} \quad r, s \in \mathbb{T}^k$$

Remark 7. It can be shown that every rd-continuous function has an antiderivative.

Example 8. If $\mathbb{T}=\mathbb{R}$, then $\int\limits_a^b f(\tau)\triangle\tau = \int\limits_a^b f(\tau)d\tau$, where the integral on the right is the usual Riemann integral.

If $\mathbb{T}=\mathbb{Z}$, then $\int\limits_a^b f(\tau)\triangle\tau = \sum\limits_{t=a}^{b-1} f(t)$ for $a<b$.

If $\mathbb{T}=h\mathbb{Z}$, $h>0$, then $\int\limits_a^b f(\tau)\triangle\tau = \sum\limits_{t=\frac{a}{h}}^{\frac{b}{h}-1} f(th)h$ for $a<b$.

Proposition 9.

$$\int_a^b f(t)g^{\triangle}(t)\triangle t =$$

$$(fg)(b) - (fg)(a) - \int_a^b f^{\triangle}(t)g(\sigma(t))\triangle t.$$

If f is rd-continuous and $t \in \mathbb{T}^k$, then

$$\int_t^{\sigma(t)} f(\tau)\triangle(\tau) = \mu(t)f(t).$$

If $f(t) \geq 0$ for all $a \leq t < b$, then

$$\int_a^b f(\tau)\triangle(\tau) \geq 0.$$

Differential equations on time scales

An $n \times n$ matrix-valued function A on $\mathbb{T}$ is called *regressive* with respect to $\mathbb{T}$ provided $I + \mu(t)A(t)$ is invertible for all $t \in \mathbb{T}^k$.

The system of delta differential equations

$$x^{\triangle}(t) = A(t)x(t)$$

is called *regressive* provided A is regressive.

Remark 10. If $\mathbb{T} = \mathbb{R}$, then any matrix-valued function A on $\mathbb{T}$ satisfies the regressivity condition. Then we get standard differential equations

$$\dot{x}(t) = A(t)x(t).$$

If $\mathbb{T} = \mathbb{Z}$, then any matrix-valued function A on $\mathbb{T}$ is rd-continuous. In order that A be regressive, the matrix $I + A(t)$ needs to be invertible for each $t \in \mathbb{Z}$. Then we get

$$x(t+1) - x(t) = A(t)x(t)$$

or

$$x(t+1) = x(t) + A(t)x(t) = (I + A(t))x(t).$$

Theorem 11. *Let A be regressive and rd-continuous $n \times n$ matrix-valued function on $\mathbb{T}$. Then the initial value problem*

$$x^{\triangle} = A(t)x, \quad x(t_0) = x_0$$

has a unique solution x defined on $\mathbb{T}$.

Let $t_0 \in \mathbb{T}$ and let A be regressive and rd-continuous $n \times n$ matrix-valued function. The unique matrix-valued solution of the initial value problem $X^{\triangle} = A(t)X$, $X(t_0) = I$, is called the *matrix exponential function of A* (at t_0). Its value at $t \in \mathbb{T}$ will be denoted by $e_A(t, t_0)$

Remark 12. Let A be a constant $n \times n$ matrix. If $\mathbb{T} = \mathbb{R}$, then $e_A(t, t_0) = e^{A(t-t_0)}$.
If $\mathbb{T} = \mathbb{Z}$ and $I + A$ is invertible, then $e_A(t, t_0) = (I + A)^{(t-t_0)}$.

Remark 13. If A is not regressive we still have unique forward solutions of

$$x^{\triangle} = A(t)x, \quad x(t_0) = x_0$$

defined for all $t \geq t_0$.

Consider the nonhomogeneous equation

$$x^{\triangle} = A(t)x + f(t), \quad x(t_0) = x_0 \qquad (1)$$

where $f : \mathbb{T} \to \mathbb{R}^n$ is a vector-valued rd-continuous function, $t_0 \in \mathbb{T}$ and $x_0 \in \mathbb{R}^n$.

Theorem 14. *Let A be a rd-continuous regressive $n \times n$ matrix-valued function on $\mathbb{T}$. Then the initial value problem (1) has a unique solution on $\mathbb{T}$, given by*

$$x(t) = e_A(t, t_0)x_0 + \int_{t_0}^{t} e_A(t, \sigma(\tau))f(\tau)\triangle\tau$$

Control systems on time scales – selected examples

Controllability and observability

Consider a linear time-variant control system with output

$$
\begin{aligned}
x^{\triangle}(t) &= A(t)x(t) + B(t)u(t) \\
y(t) &= C(t)x(t) + D(t)u(t)
\end{aligned}
\tag{2}
$$

with the initial condition $x(t_0) = x_0$. We assume that $t \in \mathbb{T}$, $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, $y(t) \in \mathbb{R}^p$ and A, B, C, D are rd-continuous, A is regressive and controls u are piecewise rd-continuous.

Notation: for $t_0, t_1 \in \mathbb{T}$, $[t_0, t_1] = \{t \in \mathbb{T} : t_0 \leq t \leq t_1\}$.

We say that system (2) is *controllable* on $[t_0, t_1] \subseteq \mathbb{T}$ if any state $x \in \mathbb{R}^n$ can be reached from any other state starting at time t_0 and finishing at time t_1, using piecewise constant controls.

Two states are *indistinguishable* on $[t_0, t_1]$ if trajectories starting at those points and corresponding to the same control give rise to the same output on $[t_0, t_1]$. The systems is *observable* on $[t_0, t_1]$ if it does not have distinct states indistinguishable on $[t_0, t_1]$.

Let us consider the Gramian controllability matrix

$$W(t_0, t_1) = \int_{t_0}^{t_1} e_A(t_1, \sigma(\tau)) B(\tau) B^T(\tau) e_A^T(t_1, \sigma(\tau)) \triangle\tau$$

Theorem 15. *The system (2) is controllable on $[t_0, t_1]$ iff the matrix $W(t_0, t_1)$ is nonsingular.*

Now let us consider the Gramian observability matrix

$$M(t_0, t_1) = \int_{t_0}^{t_1} e_A^T(\tau, t_0) C^T(\tau) C(\tau) e_A(\tau, t_0) \triangle\tau$$

Theorem 16. *The system (2) is observable on $[t_0, t_1]$ iff the matrix $M(t_0, t_1)$ is nonsingular.*

Theorem 17. *Let* $[t_0, t_1]$ *contain at least* $n+1$ *elements and* A, B, C, D *be constant matrices. The system (2) is controllable on* $[t_0, t_1]$ *if and only if*

$$\text{rank}(B, AB, \ldots, A^{n-1}B) = n.$$

The system (2) is observable on $[t_0, t_1]$ *if and only if*

$$\text{rank}\begin{pmatrix} C \\ CA \\ \ldots \\ CA^{n-1} \end{pmatrix} = n.$$

Dynamical equivalence

Assumptions: the time scale $\mathbb{T}$ is *homogeneous*, i.e. $t+\mathbb{T} := \{t+s, s \in \mathbb{T}\} = \mathbb{T}$ for every $t \in \mathbb{T}$, and $0 \in \mathbb{T}$.

Consider the following control system Σ defined on $\mathbb{T}$

$$x^{\triangle}(t) = f(x(t), u(t)), \qquad (3)$$

where $t \in \mathbb{T}$, $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$ and f is a map from $\mathbb{R}^n \times \mathbb{R}^m$ into $\mathbb{R}^n$. We assume that f and u are of class C^∞.

A *trajectory* of Σ is any pair (x, u) of functions defined on a subset of $\mathbb{T}$ that satisfy (3). The *behavior* of Σ, denoted by $\mathcal{B}(\Sigma)$, is the set of all its trajectories.

Assumptions on the system Σ:

Condition A. For every $x, y \in \mathbb{R}^n$ there is at most one u that satisfies the equation

$$y = f(x, u). \tag{4}$$

Condition B. For any x and u the rank of the matrix

$$\frac{\partial f}{\partial u}(x, u)$$

is full (i.e. equal m).

Condition C. The map $\mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n \times \mathbb{R}^n$: $(x, u) \mapsto (x, f(x, u))$ is proper, i.e. the inverse image of a compact set in $\mathbb{R}^n \times \mathbb{R}^n$ is a compact set in $\mathbb{R}^n \times \mathbb{R}^m$.

Let us consider two systems

$$\Sigma: \; x^{\triangle}(t) = f(x(t), u(t))$$

and

$$\tilde{\Sigma}: \; \tilde{x}^{\triangle}(t) = \tilde{f}(\tilde{x}(t), \tilde{u}(t))$$

with $x(t) \in \mathbb{R}^n$, $\tilde{x}(t) \in \mathbb{R}^{\tilde{n}}$, $u(t), \tilde{u}(t) \in \mathbb{R}^m$, defined on the same time scale $\mathbb{T}$.

Consider dynamic feedback transformations of one system into the other and vice versa

$$\begin{aligned}
x(t) &= \phi(\tilde{x}(t), \ldots, \tilde{x}^{(r)}(t)), \\
u(t) &= \psi(\tilde{x}(t), \ldots, \tilde{x}^{(r)}(t), \tilde{u}(t), \ldots, \tilde{u}^{(r)}(t)) \\
\tilde{x}(t) &= \tilde{\phi}(x(t), \ldots, x^{(r)}(t)), \\
\tilde{u}(t) &= \tilde{\psi}(x(t), \ldots, x^{(r)}(t), u(t), \ldots, u^{(r)}(t))
\end{aligned}$$

We say that two systems Σ and $\tilde{\Sigma}$ are *dynamically feedback equivalent* if there are dynamic feedback transformations that transform the behavior of one system onto the behavior of the second system and vice versa, and these transformations are mutually inverse on the behaviors.

A nonlinear control system is *dynamically feedback linearizable* if it is dynamically feedback equivalent to a linear controllable system.

Let $J(m)$ denote the space of all infinite sequences $U = (u_0, u_1, \ldots)$, where $u_k \in \mathbb{R}^m$. Let $A(n, m)$ denote the algebra of all C^∞ functions

$$\varphi : \mathbb{R}^n \times J(m) \to \mathbb{R}$$

depending only on a finite number of elements in $U \in J(m)$. Let us now consider a system Σ, described by (3). Define the operator $\delta_\Sigma : A(n, m) \to A(n, m)$ associated with Σ by

$$\begin{aligned} &(\delta_\Sigma \varphi)(x, U) := \\ &\int_0^1 \frac{\partial \varphi}{\partial x}(x + h\mu(0) f(x, u_0), U) dh \cdot f(x, u_0) + \\ &\sum_{i=0}^{\infty} \int_0^1 \frac{\partial \varphi}{\partial u_i}(x, U + h\mu(0) U_1)) dh \cdot u_{i+1}. \end{aligned} \tag{5}$$

The algebra $A(n, m)$ together with the operator δ_Σ is called the *delta algebra of system* Σ and denoted by A_Σ. A homomorphism of delta algebras A_Σ and $A_{\tilde{\Sigma}}$ is a homomorphism $\tau : A(n, m) \to A(\tilde{n}, m)$ of algebras that satisfies the condition $\delta_{\tilde{\Sigma}} \circ \tau = \tau \circ \delta_\Sigma$. An *isomorphism* of the delta algebras A_Σ and $A_{\tilde{\Sigma}}$ is a homomorphism that is a bijective map.

Theorem 18. *Systems Σ and $\tilde{\Sigma}$ are dynamically feedback equivalent if and only if their delta algebras A_Σ and $A_{\tilde{\Sigma}}$ are isomorphic.*

Remark 19. If the time scale is the real line, then the delta algebra of the system becomes the differential algebra of the continuous-time system (Jakubczyk, 1993). On the other hand, if the time scale is the set of integer numbers, then the delta algebra of the system is related to the difference algebra of the discrete-time system (Bartosiewicz, Jakubczyk, Pawłuszewicz, 1994).

Stability of linear systems on homogeneous time scales

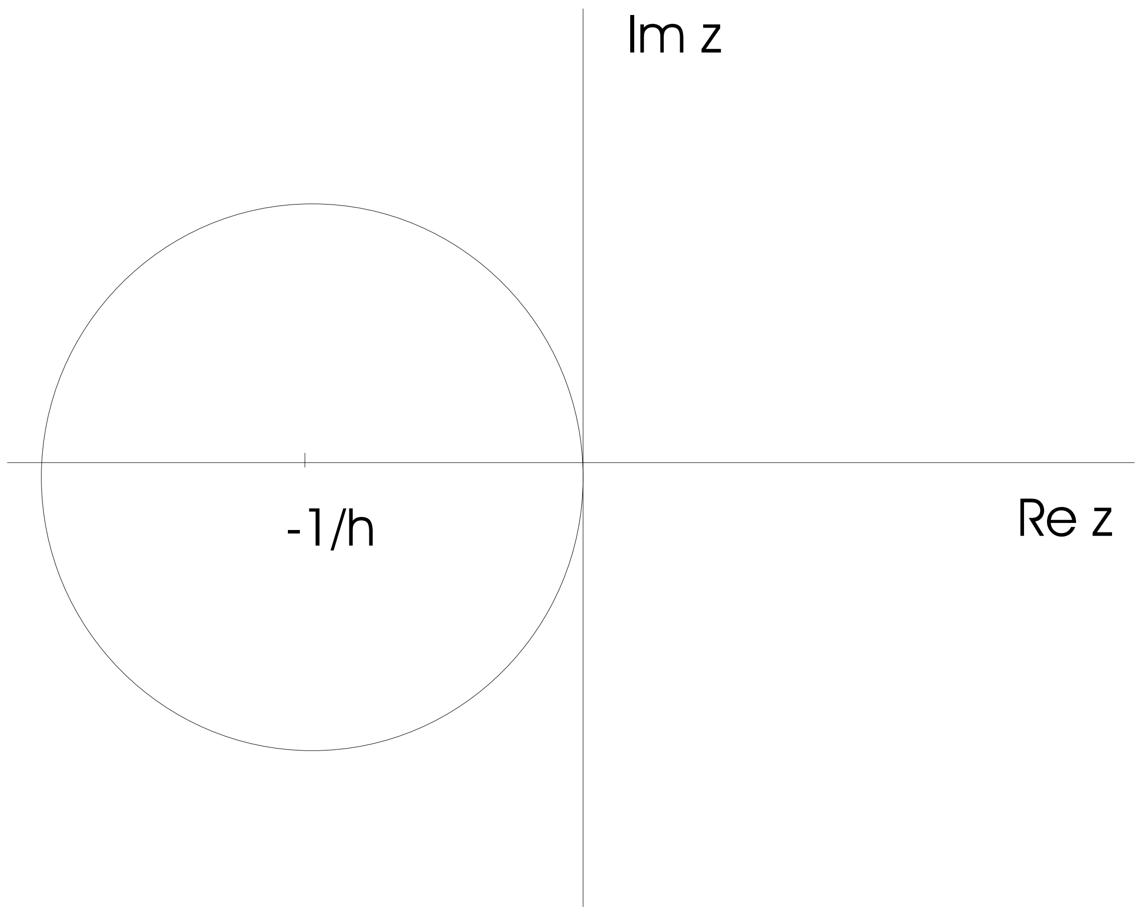
Consider the system

$$x^{\triangle} = Ax \tag{6}$$

where $\mathbb{T} = \mathbb{R}$ or $\mathbb{T} = h\mathbb{Z}$ and A is $n \times n$ constant matrix.

System (6) is *asymptotically stable* if for any initial condition the solution goes to 0 when $t \to +\infty$.

Proposition 20. *System (6) is asymptotically stable if and only if all eigenvalues of A lie in the open disc with the center at $\frac{-1}{h}$ and the radius $\frac{1}{h}$. For $\mathbb{T} = \mathbb{R}$ take the limit with $h \to 0$.*


Im z
-1/h
Re z

Concluding remarks

- Time scale is a model of time. Time may be continuous, discrete or mixed.

- Calculus on time scales unifies differential and difference calculus. Differential equations and difference equations are particular cases of differential equations on time scales.

- Control theory on time scales unifies theories of continuous-time systems and discrete-time systems, and allows to study systems with mixed time.

- Virtually any topic in control theory may be rewritten in the language of time scales.

- Systems on double time scales may allow to model hybrid systems, where some variables depend on continuous time and other on discrete time.